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An exact integral equation for the renormalized Fermi surface

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Abstract

The true Fermi surface of a fermionic many-body system can be viewed as a fixed point manifold of the renormalization group (RG). Within the framework of the exact functional RG we show that the fixed point condition implies an exact integral equation for the counter-term which is needed for a self-consistent calculation of the Fermi surface. In the simplest approximation, our integral equation reduces to the self-consistent Hartree–Fock equation for the counter-term.

1. Introduction

In his authoritative book on interacting Fermi systems, Nozières wrote 40 years ago [1]: 'In practice, we shall never try to *calculate* the Fermi surface, which is much too difficult'. What is the reason for this difficulty? Formally, the Fermi surface of an interacting Fermi system is defined as the set of all wavevectors $\boldsymbol{k}_F$ satisfying [2]

$$\epsilon_{\boldsymbol{k}_F} - \mu + \Sigma(\boldsymbol{k}_F, \mathrm{i}0) = 0, \tag{1}$$

where $\epsilon_{\boldsymbol{k}}$ is the energy dispersion in the absence of interactions, μ is the chemical potential, and $\Sigma(\boldsymbol{k}, \omega)$ is the exact self-energy of the interacting system[1]. For simplicity we assume an infinite and spin-rotationally invariant system at zero temperature, so that $\Sigma(\boldsymbol{k}, \omega)$ is independent of the spin. Unfortunately, the function $\Sigma(\boldsymbol{k}_F, \mathrm{i}0)$ in equation (1) is not known *a priori*, so the calculation of the true Fermi surface requires the solution of the many-body problem.

For weak interactions, one might try to determine the Fermi surface perturbatively by simply calculating $\Sigma(\boldsymbol{k}, \mathrm{i}0)$ in powers of the interaction and substituting the result into equation (1). However, in general the perturbation series contains anomalous terms [3] with unphysical singularities, which are generated because the *ground state* of the non-interacting system evolves into an *excited state* of the interacting system when the interaction

[1] We assume that $\Sigma(\boldsymbol{k}_F, \mathrm{i}0)$ is real, which is always true for Fermi liquids, where the damping of quasiparticles with wavevectors precisely on the Fermi surface vanishes.

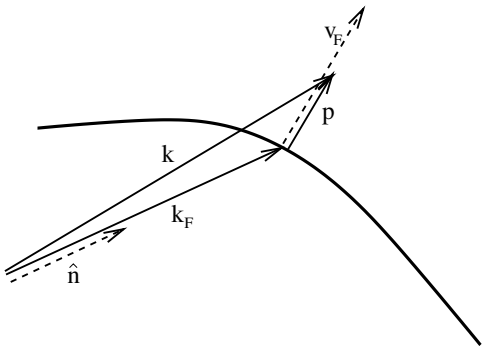

Figure 1. The decomposition $\boldsymbol{k} = \hat{\boldsymbol{n}} k_F(\hat{\boldsymbol{n}}) + \boldsymbol{p}$ of a wavevector $\boldsymbol{k}$ into a component $\boldsymbol{k}_F = \hat{\boldsymbol{n}} k_F(\hat{\boldsymbol{n}})$ on the Fermi surface and a component $\boldsymbol{p}$ in the direction of the local Fermi velocity $\boldsymbol{v}_F$. The thick solid curve is a part of the Fermi surface. This construction defines $\boldsymbol{k}_F$ and $\hat{\boldsymbol{n}}$ as a function of $\boldsymbol{k}$. Here $\boldsymbol{v}_F = \boldsymbol{\nabla}_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}}|_{\boldsymbol{k}=\boldsymbol{k}_F}$ is defined in terms of the gradient of the free dispersion at the true Fermi surface, so $\boldsymbol{v}_F$ is not necessarily perpendicular to the Fermi surface.

is adiabatically switched on. As discussed by Nozières [1], this artificial level crossing can be avoided by introducing counter-terms which are determined by the requirement that the Fermi surface remains fixed as the interaction is adiabatically switched on. This intuitive idea can be implemented perturbatively as follows [1, 4]: suppose we would like to know the true Fermi surface of a system with Hamiltonian $H = H_0 + H_1$, where H_1 describes some general two-body interaction and the non-interacting part is given by

$$H_0 = \sum_{\boldsymbol{k},\sigma} \epsilon_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma}. \tag{2}$$

Here $c_{\boldsymbol{k}\sigma}$ are the usual annihilation operators of fermions with momentum $\boldsymbol{k}$ and spin σ. An expansion in powers of H_1 leads to Feynman diagrams where vertices corresponding to H_1 are connected by propagators $G_0(\boldsymbol{k}, \omega) = [\omega - \epsilon_{\boldsymbol{k}} + \mu]^{-1}$. These are singular for $\omega = 0$ and $\epsilon_{\boldsymbol{k}} = \mu$, which is not the true Fermi surface defined in equation (1). If the perturbative expansion is truncated at a finite order, this leads to the unphysical divergences mentioned above [3]. To avoid these, we add the counter-term $\sum_{\boldsymbol{k}\sigma} \Sigma(\boldsymbol{k}_F, \mathrm{i}0) c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma}$ to H_0 and subtract it again from H_1, writing $H = H_0' + H_1'$, with

$$H_0' = \sum_{\boldsymbol{k},\sigma} [\epsilon_{\boldsymbol{k}} + \Sigma(\boldsymbol{k}_F, \mathrm{i}0)] c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma}, \tag{3}$$

and $H_1' = H_1 - \sum_{\boldsymbol{k}\sigma} \Sigma(\boldsymbol{k}_F, \mathrm{i}0) c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma}$. Here $\boldsymbol{k}_F$ is the wavevector closest to $\boldsymbol{k}$ lying on the Fermi surface; see figure 1. Using equation (1), the corresponding free propagator can then be written as $G_0'(\boldsymbol{k}, \omega) = [\omega - \epsilon_{\boldsymbol{k}} + \epsilon_{\boldsymbol{k}_F}]^{-1}$, which by construction is singular on the true Fermi surface. But how do we find the counter-term necessary for calculating $\epsilon_{\boldsymbol{k}_F}$? Following the usual strategy adopted in field theory [5], we may expand the irreducible self-energy associated with the modified interaction H_1' perturbatively in powers of H_1' and require that, order by order in perturbation theory, the corrections vanish when we set $\omega = 0$ and $\boldsymbol{k} = \boldsymbol{k}_F$. This renormalized perturbation theory leads to complicated integral equations for $\Sigma(\boldsymbol{k}_F, \mathrm{i}0)$, which must be solved numerically [6].

A few years ago Anderson [7] critically discussed a simplified version of the renormalized perturbation theory outlined above, where $\Sigma(\boldsymbol{k}_F, \mathrm{i}0)$ is replaced by a constant $\delta\mu$. He correctly pointed out that in general it is not permissible to ignore the momentum dependence of the counter-terms, and argued that in two dimensions the effective renormalized interaction

between electrons with opposite spin and wavevectors on the true Fermi surface depends in a subtle way on the boundary conditions, which cannot be adequately taken into account perturbatively. Thus, according to Anderson a non-perturbative calculation of the renormalized two-body interaction between strongly correlated electrons in two dimensions is a prerequisite to understanding the physics of these systems. One of the key points of [7] is the insight that without a non-perturbative method of locating the true Fermi surface it is not possible to calculate the correct effective interaction. As a small step towards the solution of this problem, we offer in the present work an algorithm for calculating the true Fermi surface which does not rely on the perturbative expansion of counter-terms in powers of the interaction. We show that such an algorithm follows in a straightforward way from the exact functional renormalization group (RG) approach described in [8].

2. The Fermi surface as a fixed point manifold of the renormalization group

In the past decade several authors have used Wilsonian RG methods to study interacting Fermi systems [8–15], using different versions of the RG. In particular, the RG transformations used in [12–15] exclusively focus on the mode elimination step. Although such a procedure is sufficient if one considers only the one-loop flow of marginal couplings, in general a complete Wilsonian RG transformation consists not only of the mode elimination, but includes also the rescaling of momenta, frequencies, and fields [16]. While the field rescaling (i.e. the wavefunction renormalization) only becomes important beyond the one-loop approximation, the RG flow of all relevant and irrelevant couplings is determined in an essential way by the rescaling step even at the one-loop level. We emphasize that the rescaling of momenta and frequencies is more than a trivial mathematical change of variables—it is crucial for detecting possible fixed points of the RG and calculating critical exponents [16]. Moreover, as will be shown below, the rescaling of momenta and frequencies is very helpful for a derivation of the self-consistent Hartree–Fock approximation within the framework of the exact RG. The importance of rescaling in the presence of a Fermi surface has been emphasized by Polchinski [10] and by Shankar [11]. In [8] we have shown how the exact functional RG approach developed previously [12–14] can be modified to include the rescaling step. Here, we shall show that the rescaling is crucial for obtaining a non-perturbative definition of the Fermi surface as a fixed point manifold of the RG. We obtain an explicit algorithm for calculating the Fermi surface which does not rely on the iterative procedure of fixing the counter-terms order by order in perturbation theory. The fixed point property of the Fermi surface has also been emphasized by Ferraz [17], who recently discussed the Fermi surface renormalization in a special two-dimensional system using the field theoretical RG.

The exact functional RG can be formulated in terms of an infinite hierarchy of coupled differential equations for the irreducible $2n$-point vertices $\Gamma_\xi^{(2n)}(K_1', \ldots, K_n'; K_n, \ldots, K_1)$, where $K = (\sigma, \boldsymbol{k}, \mathrm{i}\omega)$ is a collective label for spin projection σ, momentum $\boldsymbol{k}$, Matsubara frequency $\mathrm{i}\omega$. Here ξ is an infrared cut-off with units of energy which regularizes the singularity of the free propagator,

$$G'_{\xi,\xi_0}(K) = \frac{\Theta(\Omega_K - \xi) - \Theta(\Omega_K - \xi_0)}{\mathrm{i}\omega_n - \epsilon_{\boldsymbol{k}} + \epsilon_{\boldsymbol{k}_F}}, \tag{4}$$

where $\Omega_K = |\epsilon_{\boldsymbol{k}} - \epsilon_{\boldsymbol{k}_F}|$. To determine the true Fermi surface self-consistently it turns out to be useful to subtract the counter-term $\Sigma(\boldsymbol{k}_F, \mathrm{i}0)$ from the irreducible two-point vertex [8], defining

$$\Gamma_\xi^{(2)}(K) = -\Sigma_\xi(K) + \Sigma(\boldsymbol{k}_F, \mathrm{i}0), \tag{5}$$

where $\Sigma_\xi(\boldsymbol{k}, \mathrm{i}\omega)$ is the irreducible self-energy of the system with infrared cut-off ξ. The exact flow equation for the vertex $\Gamma_\xi^{(2)}(K)$ can be written in the form

$$\partial_\xi \Gamma_\xi^{(2)}(K) = \int_{K'} \frac{\delta(\Omega_{K'} - \xi)}{\mathrm{i}\omega_{n'} - \epsilon_{\boldsymbol{k}'} + \epsilon_{\boldsymbol{k}'_F} + \Gamma_\xi^{(2)}(K')} \Gamma_\xi^{(4)}(K, K'; K', K), \tag{6}$$

where $\int_K = \sum_\sigma \int \frac{\mathrm{d}^D k}{(2\pi)^D} \int \frac{\mathrm{d}\omega}{2\pi}$ in D dimensions. The exact flow equation for the irreducible four-point vertex $\Gamma_\xi^{(4)}$ on the right-hand side of equation (6) involves in turn the irreducible six-point vertex $\Gamma_\xi^{(6)}$,

$$\begin{aligned}
\partial_\xi \Gamma_\xi^{(4)}(K_1', K_2'; K_2, K_1) = {} & \int_K \frac{\delta(\Omega_K - \xi)}{\mathrm{i}\omega_n - \epsilon_{\boldsymbol{k}} + \epsilon_{\boldsymbol{k}_F} + \Gamma_\xi^{(2)}(K)} \Gamma_\xi^{(6)}(K_1', K_2', K; K, K_2, K_1) \\
& + \int_K \left[\frac{\delta(\Omega_K - \xi) G_{\xi,\xi_0}(K')}{\mathrm{i}\omega_n - \epsilon_{\boldsymbol{k}} + \epsilon_{\boldsymbol{k}_F} + \Gamma_\xi^{(2)}(K)} + \frac{G_{\xi,\xi_0}(K) \delta(\Omega_{K'} - \xi)}{\mathrm{i}\omega_{n'} - \epsilon_{\boldsymbol{k}'} + \epsilon_{\boldsymbol{k}'_F} + \Gamma_\xi^{(2)}(K')} \right] \\
& \times \Big\{ \tfrac{1}{2} \big[\Gamma_\xi^{(4)}(K_1', K_2'; K', K) \Gamma_\xi^{(4)}(K, K', K_2, K_1) \big]_{K' = K_1 + K_2 - K} \\
& - \big[\Gamma_\xi^{(4)}(K_1', K'; K, K_1) \Gamma_\xi^{(4)}(K_2', K; K', K_2) \big]_{K' = K + K_1 - K_1'} \\
& + \big[\Gamma_\xi^{(4)}(K_2', K'; K, K_1) \Gamma_\xi^{(4)}(K_1', K; K', K_2) \big]_{K' = K + K_1 - K_2'} \Big\},
\end{aligned} \tag{7}$$

where

$$G_{\xi,\xi_0}(K) = \frac{\Theta(\Omega_K - \xi) - \Theta(\Omega_K - \xi_0)}{\mathrm{i}\omega_n - \epsilon_{\boldsymbol{k}} + \epsilon_{\boldsymbol{k}_F} + \Gamma_\xi^{(2)}(K)}. \tag{8}$$

The flow equation for the six-point vertex is given in the appendix of [8].

To scale wavevectors toward the Fermi surface, it is convenient to perform a non-linear coordinate transformation in momentum space, $\boldsymbol{k} = \hat{\boldsymbol{n}} k_F(\hat{\boldsymbol{n}}) + \hat{\boldsymbol{v}}_F \xi q / |\boldsymbol{v}_F|$, and eliminate $\boldsymbol{k}$ in favour of the dimensionless variable q and the unit vector $\hat{\boldsymbol{n}}$. Here $k_F(\hat{\boldsymbol{n}})$ is the length of $\boldsymbol{k}_F$ parametrized by $\hat{\boldsymbol{n}}$, and $\boldsymbol{v}_F = \boldsymbol{\nabla}_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}}|_{\boldsymbol{k}_F}$ is the Fermi velocity of the non-interacting system at the true $\boldsymbol{k}_F$; see figure 1. The corresponding unit vector is denoted by $\hat{\boldsymbol{v}}_F = \boldsymbol{v}_F / |\boldsymbol{v}_F|$. Geometrically, $q = \boldsymbol{v}_F \cdot \boldsymbol{p}/\xi = \boldsymbol{v}_F \cdot (\boldsymbol{k} - \hat{\boldsymbol{n}} k_F(\hat{\boldsymbol{n}}))/\xi$ measures the distance of a given $\boldsymbol{k}$ from the Fermi surface in units of the infrared cut-off. We also define rescaled frequencies $\epsilon = \omega/\xi$, and label the degrees of freedom by $Q = (\sigma, \hat{\boldsymbol{n}}, q, \mathrm{i}\epsilon)$ instead of K. To implement the *scaling toward the Fermi surface* [10] we consider the RG flow of the following rescaled vertices [8],

$$\tilde{\Gamma}_t^{(2)}(Q) = \frac{Z_t^{\hat{n}}}{\xi} \Gamma_\xi^{(2)}(K) = -\frac{Z_t^{\hat{n}}}{\xi} [\Sigma_\xi(K) - \Sigma(\boldsymbol{k}_F, \mathrm{i}0)], \tag{9}$$

$$\begin{aligned}
\tilde{\Gamma}_t^{(2n)}(Q_1', \ldots, Q_n'; Q_n, \ldots, Q_1) & \\
= \nu_0^{n-1} \xi^{n-2} [Z_t^{\hat{n}_1'} \cdots Z_t^{\hat{n}_n'} Z_t^{\hat{n}_n} \cdots Z_t^{\hat{n}_1}]^{1/2} \Gamma_\xi^{(2n)}(K_1', \ldots, K_n'; K_n, \ldots, K_1). &
\end{aligned} \tag{10}$$

Here $t = -\ln(\xi/\xi_0)$ is a logarithmic flow parameter, $\nu_0 = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^D} \delta(\epsilon_{\boldsymbol{k}} - \epsilon_{\boldsymbol{k}_F})$ is the density of states (per spin projection) of the non-interacting system at the true Fermi surface, and

$$Z_t^{\hat{n}} = \left[1 - \frac{\partial \Sigma_\xi(\boldsymbol{k}_F, \omega + \mathrm{i}0)}{\partial \omega} \bigg|_{\omega = 0} \right]^{-1} \tag{11}$$

is the wavefunction renormalization factor.

From equation (6) and the above definitions, it is now easy to show that our rescaled two-point vertex satisfies the exact flow equation [8]

$$\partial_t \tilde{\Gamma}_t^{(2)}(Q) = (1 - \eta_t^{\hat{n}} - q \partial_q - \epsilon \partial_\epsilon) \tilde{\Gamma}_t^{(2)}(Q) - \int_{Q'} \dot{G}_t(Q') \tilde{\Gamma}_t^{(4)}(Q, Q'; Q', Q), \tag{12}$$

where $\eta_t^{\hat{n}} = -\partial_t \ln Z_t^{\hat{n}}$ is the flowing anomalous dimension (which vanishes for large t if the system is a Fermi liquid). The integration measure in equation (12) is defined by

$$\int_Q = \sum_\sigma \int \frac{\mathrm{d}S_{\hat{n}}}{S_D} \int \mathrm{d}q\, J(\hat{n}, q) \int \frac{\mathrm{d}\epsilon}{2\pi}, \tag{13}$$

where $\mathrm{d}S_{\hat{n}}$ is a surface element and S_D is the surface area of the unit sphere in D dimensions, and $J(\hat{n}, q)$ is the Jacobian associated with the transformation $\boldsymbol{k} \to (\hat{n}, q)$ divided by $\nu_0 \xi^2$. The function $\dot{G}_t(Q)$ in equation (12) is defined by

$$\dot{G}_t(Q) = \frac{\delta(\tilde{\Omega}_Q - 1)}{Z_t^{\hat{n}}\big[\mathrm{i}\epsilon - \xi_t^{\hat{n}}(q)\big] + \tilde{\Gamma}_t^{(2)}(Q)}, \tag{14}$$

where $\xi_t^{\hat{n}}(q) = (\epsilon_{k_t} - \epsilon_{k_F})/\xi$, with $\boldsymbol{k}_t = \hat{\boldsymbol{n}} k_F(\hat{\boldsymbol{n}}) + \hat{\boldsymbol{v}}_F \xi_0 \mathrm{e}^{-t} q/|\boldsymbol{v}_F|$. Here $\tilde{\Omega}_Q = |\xi_t^{\hat{n}}(q)| \approx |q|$. The flow equation of the rescaled irreducible four-point vertex $\tilde{\Gamma}_t^{(4)}(Q_1', Q_2'; Q_2, Q_1)$ that follows from equation (7) in a straightforward way is not explicitly needed in this work; it can be found in [8].

The shape of the Fermi surface of the interacting system is determined by the RG flow of the couplings

$$\tilde{\mu}_t^{\hat{n}} \equiv \tilde{\Gamma}_t^{(2)}(\sigma, \hat{\boldsymbol{n}}, q = 0, \mathrm{i}\epsilon = \mathrm{i}0). \tag{15}$$

From equation (12) we see that these couplings satisfy the exact flow equation

$$\partial_t \tilde{\mu}_t^{\hat{n}} = (1 - \eta_t^{\hat{n}}) \tilde{\mu}_t^{\hat{n}} + \dot{\Gamma}_t^{(2)}(\hat{\boldsymbol{n}}), \tag{16}$$

with

$$\dot{\Gamma}_t^{(2)}(\hat{\boldsymbol{n}}) = -\int_{Q'} \dot{G}_t(Q') \tilde{\Gamma}_t^{(4)}(Q_0, Q'; Q', Q_0), \tag{17}$$

where $Q_0 = (\sigma, \hat{\boldsymbol{n}}, q = 0, \mathrm{i}\epsilon = \mathrm{i}0)$. Note that for dimensions $D > 1$ there are infinitely many couplings $\tilde{\mu}_t^{\hat{n}}$, labelled by the unit vector $\hat{\boldsymbol{n}}$. Obviously, each $\tilde{\mu}_t^{\hat{n}}$ with $\eta_\infty^{\hat{n}} = \lim_{t\to\infty} \eta_t^{\hat{n}} < 1$ is relevant, so some fine tuning of the bare couplings $\tilde{\mu}_0^{\hat{n}}$ is necessary to force $\tilde{\mu}_t^{\hat{n}}$ to flow into a fixed point of the RG. Because a finite limit $\tilde{\mu}_\infty^{\hat{n}} = \lim_{t\to\infty} \tilde{\mu}_t^{\hat{n}}$ means that we have found the true Fermi surface of the interacting system [8], we conclude that the detailed shape of the Fermi surface is sensitive to the numerical values of the bare couplings of the theory. Polchinski [10] pointed out that such a fine tuning of the bare couplings is *unnatural*, because physical effects which depend on the precise shape of the Fermi surface are not protected against small perturbations.

To further elucidate the relation between the relevant couplings $\tilde{\mu}_t^{\hat{n}}$ and the shape of the Fermi surface, let us explicitly derive from equation (16) a self-consistency condition for the Fermi surface. It is useful to transform equation (16) into an integral equation,

$$\tilde{\mu}_t^{\hat{n}} = \mathrm{e}^{t - \int_0^t \mathrm{d}\tau\, \eta_\tau^{\hat{n}}} \left[\tilde{\mu}_0^{\hat{n}} + \int_0^t \mathrm{d}t'\, \mathrm{e}^{-t' + \int_0^{t'} \mathrm{d}\tau\, \eta_\tau^{\hat{n}}} \dot{\Gamma}_{t'}^{(2)}(\hat{\boldsymbol{n}}) \right]. \tag{18}$$

Suppose now that we have adjusted the bare couplings such that for $t \to \infty$ the flowing couplings $\tilde{\mu}_t^{\hat{n}}$ indeed approach finite fixed point values. Assuming that the associated anomalous dimensions $\eta_\infty^{\hat{n}}$ are smaller than unity[2], we conclude from equation (18) that the limit $\tilde{\mu}_\infty^{\hat{n}} = \lim_{t\to\infty} \tilde{\mu}_t^{\hat{n}}$ can only be finite if the initial values $\tilde{\mu}_0^{\hat{n}}$ are chosen such that

$$\begin{aligned} \tilde{\mu}_0^{\hat{n}} &= -\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t + \int_0^t \mathrm{d}\tau\, \eta_\tau^{\hat{n}}} \dot{\Gamma}_t^{(2)}(\hat{\boldsymbol{n}}) \\ &= \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t + \int_0^t \mathrm{d}\tau\, \eta_\tau^{\hat{n}}} \int_{Q'} \dot{G}_t(Q') \tilde{\Gamma}_t^{(4)}(Q_0, Q'; Q', Q_0). \end{aligned} \tag{19}$$

[2] For $\eta_\infty^{\hat{n}} > 1$ the coupling $\tilde{\mu}_t^{\hat{n}}$ becomes irrelevant and approaches a constant value (even without fine tuning) provided that the other couplings flow into a fixed point. In this case the gradient of the momentum distribution $\langle c_{k\sigma}^\dagger c_{k\sigma} \rangle$ with respect to $\boldsymbol{k}$ is finite for all $\boldsymbol{k}$, so a sharp Fermi surface does not exist.

This is an implicit equation for $\tilde{\mu}_0^{\hat{n}}$, relating it to the values of the two-point vertex and the four-point vertex on the entire RG trajectory. Keeping in mind that the right-hand side of equation (19) implicitly depends on $\tilde{\mu}_t^{\hat{n}}$ and that according to equation (9)

$$\Sigma(\boldsymbol{k}_F, \mathrm{i}0) - \Sigma_{\xi_0}(\boldsymbol{k}_F, \mathrm{i}0) = \xi_0 \tilde{\mu}_0^{\hat{n}} / Z_0^{\hat{n}}, \tag{20}$$

it is obvious that equation (19) can be regarded as an integral equation for the counter-term $\Sigma(\boldsymbol{k}_F, \mathrm{i}0)$, the solution of which yields the true shape of the Fermi surface. At this point it is instructive to transform equation (19) back to unrescaled variables, choosing for simplicity the initial conditions $\Sigma_{\xi_0}(\boldsymbol{k}_F, \mathrm{i}0) = 0$ and $Z_0^{\hat{n}} = 1$. Using the above definitions we find that equation (19) is equivalent to

$$\Sigma(\boldsymbol{k}_F, \mathrm{i}0) = \sum_{\sigma'} \int \frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^D} \frac{\mathrm{d}\omega'}{2\pi} \frac{\Theta(\xi_0 - \xi_{\boldsymbol{k}'})}{\mathrm{i}\omega' - \epsilon_{\boldsymbol{k}'} + \mu - \Sigma_{\xi_{\boldsymbol{k}'}}(K')} \times \Gamma^{(4)}_{\xi_{\boldsymbol{k}'}}(\boldsymbol{k}_F, \mathrm{i}0, \sigma, \boldsymbol{k}', \mathrm{i}\omega', \sigma'; \boldsymbol{k}', \mathrm{i}\omega', \sigma', \boldsymbol{k}_F, \mathrm{i}0, \sigma), \tag{21}$$

where $\xi_{\boldsymbol{k}'} = |\epsilon_{\boldsymbol{k}'} - \epsilon_{\boldsymbol{k}'_F}|$. Note that the right-hand side of equation (21) involves the flowing self-energy and four-point vertex at the scales $\xi = \xi_{\boldsymbol{k}'}$ which depend on the distance from the true Fermi surface. We emphasize that the exact integral equation (21) and the equivalent rescaled equation (19) fix the counter-term $\Sigma(\boldsymbol{k}_F, \mathrm{i}0)$ from the requirement that for $t \to \infty$ all couplings approach finite fixed point values.

If the system is a Luttinger liquid, then the rescaled version (19) of the self-consistency equation is more convenient than the unrescaled version (21), because for a Luttinger liquid the marginal part of the four-point vertex $\Gamma^{(4)}_{\xi}$ without wavefunction renormalization factors diverges for $\xi \to 0$, while the rescaled four-point vertex $\tilde{\Gamma}^{(4)}_t$ defined in equation (10) approaches for $t \to \infty$ a finite limit. In this case the divergence of the unrescaled $\Gamma^{(4)}_{\xi}$ is cancelled by the vanishing wavefunction renormalization factors at the Luttinger liquid fixed point [18].

From our point of view, the Fermi surface is a *fixed point* manifold of the RG, so it is meaningless to talk about the RG flow of the Fermi surface. However, it is possible to define a 'flowing Fermi surface' $\boldsymbol{k}_{F,t}$ via [8]

$$\epsilon_{\boldsymbol{k}_{F,t}} - \mu + \Sigma_{\xi_0 \mathrm{e}^{-t}}(\boldsymbol{k}_{F,t}, \mathrm{i}0) = 0, \tag{22}$$

which by construction approaches the true Fermi surface for $t \to \infty$. If we choose the initial conditions at $t = 0$ (corresponding to $\xi = \xi_0$) such that $\Sigma_{\xi_0}(\boldsymbol{k}_{F,0}, \mathrm{i}0) = 0$, then $\boldsymbol{k}_{F,0}$ is the Fermi surface of the non-interacting system at the *same chemical potential* as the interacting system. This corresponds to a non-interacting system at the density $n_0(\mu) = \sum_\sigma \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^D} \Theta(k_{F,0}(\hat{\boldsymbol{n}}) - |\boldsymbol{k}|)$, which is in general different from the density $n(\mu)$ of the interacting system given in equation (23). Note that in Fermi liquid theory one usually works at constant density. However, as discussed by Nozières[3], for a self-consistent calculation of the Fermi surface it is more convenient to determine the counter-term at constant chemical potential μ, and calculate the corresponding density afterwards[4]. The practical advantages of such a procedure have been recognized previously in [19]. Moreover, in the field theoretical approach advanced by Ferraz [17], the chemical potential is also a RG invariant.

[3] See the discussion on p 237 of [1].

[4] To compare the Fermi surface of the interacting system with the corresponding Fermi surface without interactions at the same density n, we should first calculate the density $n(\mu)$ corresponding to the Fermi surface $\boldsymbol{k}_F$ of the interacting system from equation (23), and then determine μ_0 such that $n(\mu) = \sum_\sigma \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^D} \Theta(\mu_0 - \epsilon_{\boldsymbol{k}})$. Given μ_0, we may calculate the Fermi surface of the non-interacting system at the density $n(\mu)$ from the roots of $\epsilon_{\boldsymbol{k}} = \mu_0$.

3. Possible applications

Given a solution $\Sigma(\boldsymbol{k}_F, \mathrm{i}0)$ of equation (21), we may calculate the compressibility of the system by substituting the result for $\Sigma(\boldsymbol{k}_F, \mathrm{i}0)$ into equation (1) and solving for $k_F(\hat{\boldsymbol{n}})$, which implicitly depends on the chemical potential μ. According to the Luttinger theorem [2] the density $n(\mu)$ of the system is determined by the volume enclosed by the Fermi surface,

$$n(\mu) = \sum_\sigma \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^D}\, \Theta(k_F(\hat{\boldsymbol{n}}) - |\boldsymbol{k}|), \tag{23}$$

so we obtain for the compressibility χ_n

$$n^2 \chi_n = \frac{\partial n}{\partial \mu} = \sum_\sigma \int \frac{\mathrm{d}S_{\hat{\boldsymbol{n}}}}{(2\pi)^D}\, k_F^{D-1}(\hat{\boldsymbol{n}}) \frac{\partial k_F(\hat{\boldsymbol{n}})}{\partial \mu}. \tag{24}$$

Note that the compressibility is a functional of the shape of the true Fermi surface of the many-body system.

It is instructive to apply the above procedure to a simple model of spinless fermions with linearized energy dispersion in one dimension. In the absence of scattering processes involving large momentum transfers, the marginal part of the four-point vertex can then be parametrized by a single coupling constant $\tilde{g}_t$ with a vanishing β-function [11, 18], so that $\tilde{g}_t = \tilde{g}_0$ for all t. Within the one-loop approximation the flow equation for the relevant coupling $\tilde{\mu}_t$ is then simply $\partial_t \tilde{\mu}_t = \tilde{\mu}_t - \tilde{g}_0/2$, so we obtain from equation (20) for the counter-term $\Sigma(k_F, \mathrm{i}0) = \xi_0 \tilde{g}_0/2$. For simplicity we have assumed that $\Sigma_{\xi_0}(k_F, \mathrm{i}0) \approx 0$ and $Z_0 \approx 1$, which is a good approximation for sufficiently large ξ_0. A reasonable choice is $\xi_0 = \epsilon_{k_F}$. Then we obtain from equation (24) $n^2 \chi_n = \nu_0/(1 + \tilde{g}_0/2)$, which amounts to an infinite resummation of bubble diagrams.

In the simplest approximation equations (19) and (21) reduce to the Hartree–Fock self-consistency equation for the counter-term $\Sigma(\boldsymbol{k}_F, \mathrm{i}0)$. To see this, we approximate the flowing four-point vertex in equation (21) as follows:

$$\begin{aligned}\Gamma^{(4)}_{\xi_{k'}}&(\boldsymbol{k}_F, \mathrm{i}0, \sigma, \boldsymbol{k}', \mathrm{i}\omega', \sigma'; \boldsymbol{k}', \mathrm{i}\omega', \sigma', \boldsymbol{k}_F, \mathrm{i}0, \sigma) \\ &\approx \Gamma^{(4)}_{\xi=0}(\boldsymbol{k}_F, \mathrm{i}0, \sigma, \boldsymbol{k}'_F, \mathrm{i}0, \sigma'; \boldsymbol{k}'_F, \mathrm{i}0, \sigma', \boldsymbol{k}_F, \mathrm{i}0, \sigma) \\ &\equiv \Gamma^{(4)}_0(\boldsymbol{k}_F, \sigma; \boldsymbol{k}'_F, \sigma'),\end{aligned} \tag{25}$$

i.e. we project all momenta onto the Fermi surface, ignore the frequency dependence, and replace the flowing vertex by its fixed point value. Moreover, at this level of approximation we may also make the replacement $\Sigma_{\xi_{k'}}(K') \to \Sigma(\boldsymbol{k}'_F, \mathrm{i}0)$ on the right-hand side of equation (21). For $\xi_0 \to \infty$ we then obtain the Hartree–Fock self-consistency equation

$$\Sigma(\boldsymbol{k}_F, \mathrm{i}0) = \sum_{\sigma'} \int \frac{\mathrm{d}\boldsymbol{k}'}{(2\pi)^D}\, \Gamma^{(4)}_0(\boldsymbol{k}_F, \sigma; \boldsymbol{k}'_F, \sigma') \Theta(\mu - \epsilon_{k'} - \Sigma(\boldsymbol{k}'_F, \mathrm{i}0)). \tag{26}$$

For a given interaction vertex $\Gamma^{(4)}_0(\boldsymbol{k}_F, \sigma; \boldsymbol{k}'_F, \sigma')$, equation (26) can be used to study possible Fermi surface instabilities such as the Pomeranchuk instability [20] at the Hartree–Fock level, which should always be the first step before more elaborate methods are used. A trivial generalization of equation (26) with a spin-dependent counter-term $\Sigma(\boldsymbol{k}_F, \mathrm{i}0, \sigma)$ leads to the self-consistent Hartree–Fock equation for spontaneous ferromagnetism, implying the usual Stoner instability for strong enough interactions in the spin channel. More generally, if we allow for other types of symmetry breaking, within the same approximations as above the exact RG fixed point equation for the two-point vertex can be reduced to the Hartree–Fock self-consistency equation for the corresponding order parameter. Note that no approximation

has been made in deriving equations (19) and (21), so these exact fixed point equations (or generalizations thereof for other types of symmetry breaking) can serve as a starting point for a systematic calculation of corrections to the Hartree–Fock approximation.

4. Conclusions

In summary, in this work we have shown how the true Fermi surface of an interacting Fermi system can be defined self-consistently as a fixed point property of the RG. Our main results are the two equivalent integral equations (19) and (21), which determine the counter-term $\Sigma(\boldsymbol{k}_F, \mathrm{i}0)$ necessary for calculating the Fermi surface from equation (1). In the simplest approximation, equation (21) reduces to the Hartree–Fock self-consistency condition for the counter-term. However, systematic improvements are possible.

Very recently, Dusuel and Douçot [21] presented a detailed analysis of the Fermi surface deformations in quasi-one-dimensional electronic systems, using perturbation theory and the RG method. They realized that some 'slight modification' of the Wilson–Polchinski RG approach is necessary in order to use this approach for a self-consistent calculation of the Fermi surface, but admitted that a practical implementation of such a modification remains to be attempted. We have shown here that the necessary modification of the functional RG used in [12–15] is simply the usual rescaling step [8], which is an essential part of the 'orthodox' Wilsonian RG [16]. We conclude that the functional RG approach for interacting Fermi systems in the form advanced in [8] provides an elegant solution to the difficult [1] problem of self-consistently constructing the true Fermi surface.

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